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COMBINATORIAL PROPERTIES OF POLYOMINOES

by

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A finite set of cells in the infinite planar square grid is often called a polyomino. With each polyomino P, we may associate a hypergraph whose vertices are the cells of P and whose edges are the maximal rectangles (in the standard position) contained in P. It turns out that these hypergraphs have many nice properties generalizing various properties of bipartite graphs and trees. We survey results in this direction.

0. Introduction

A finite set of cells in the infinite planar square grid is often called a polyomino [3]. With each polyomino P, we may associate a hypergraph whose vertices are the cells of P and whose edges are the maximal rectangles contained in P. (When every cell is labeled by its integer coordinates, a rectangle is a set $[a, b]\times[c, d]$ with $[r, s]$ standing for the set of integers k such that $r\leqq k\leqq s$. For notions of graph and hypergraph theory, the reader is referred to [1].) It turns out that these hypergraphs have many nice properties generalizing various properties of bipartite graphs and trees. The purpose of our article is to survey various results in this direction.

1. Covering by rectangles

Our initial interest in polyominoes was stimulated by problems in picture processing: in several different contexts, it is desirable to express a prescribed polyomino as a union of a small number of rectangles. Let $\varrho(P)$ stand for the smallest number of rectangles in P whose union is P and let $\alpha(P)$ stand for the largest size of a subset S of P such that no two cells in S lie in a common rectangle. (In a hypergraph H, a set S is called *strongly stable* if no two vertices in S lie in a common edge. The *covering number* $\varrho(H)$ is the smallest number of edges of H that cover all the vertices of H. The *strong stability* number $\alpha(H)$ is the maximum number of vertices in a strongly stable set.) Clearly $\varrho(P)\geqq\alpha(P)$ for all polyominoes P. We asked whether

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this inequality always holds with the sign of equality. S. Chaiken, D. J. Kleitman, M. Saks and J. Shearer [2] proved that this is the case whenever the polyomino P is *convex* in the sense that every horizontal line and every vertical line intersects P in an interval. On the other hand, E. Szemerédi constructed a multiply connected polyomino with $\varrho>\alpha$. Later on, F. R. K. Chung found the simply connected polyomino with $\varrho=8$ and $\alpha=7$ shown in Fig. 1.

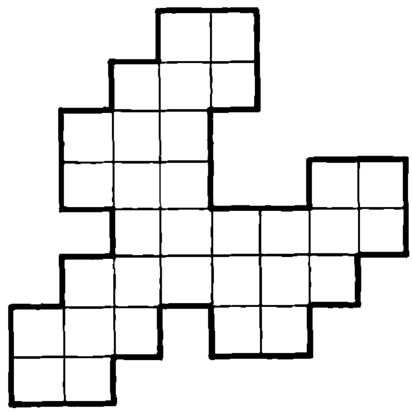

Fig. 1

We shall call a polyomino *semiconvex* if every horizontal line intersects it in an interval.

Problem 1. *Is there a semiconvex polyomino with* $\varrho>\alpha$?

W. Masek proved that computing $\varrho(P)$ is NP-hard; however, his construction produces multiply connected polyominoes.

Problem 2. *How difficult is the evaluation of* $\varrho(P)$ *and/or* $\alpha(P)$ *for simply connected polyominoes* P?

2. Packing of rectangles

The subject of the preceding section suggests a variation: let $\nu(P)$ stand for the largest number of pairwise disjoint maximal rectangles contained in P and let $\tau(P)$ stand for the smallest size of a subset of P sharing at least one cell with every maximal rectangle. (In a hypergraph H, a *matching* is a set of pairwise disjoint edges and a *transversal* is a set of vertices intersersecting every edge. The maximum cardinality of a matcing is denoted by $\nu(H)$ and the minimum cardinality of a transversal is denoted by $\tau(H)$.) Cleary $\nu(P)\leqq\tau(P)$ for all polyomioes P. We shall prove that this inequality holds with the sign of equality whenever P is semiconvex. For this purpose, we associate a certain graph $G=G(P)$ with every polyomino P. The vertices of G are the maximal rectangles in P, two such vertices being adjacent if and only if the two rectangles are disjoint.

Lemma. *If P is semiconvex then $G(P)$ is a comparability graph.*

Proof. Let $R_1=[a_1, b_1]\times[c_1, d_1]$ and $R_2=[a_2, b_2]\times[c_2, d_2]$ be maximal rectangles in a semiconvex polyomino P. If $[c_1, d_1]\subseteq[c_2, d_2]$ then we must have $[a_1, b_1]\supseteq[a_2, b_2]$: otherwise P would not be semiconvex. In particular, $[c_1, d_1]\subseteq[c_2, d_2]$ implies $R_1\cap$ $\cap R_2\neq\emptyset$. It follows that disjoint maximal rectangles R_1, R_2 have either $c_1<c_2$, $d_1<d_2$ or $c_1>c_2$, $d_1>d_2$. In the former case, we shall direct the edge R_1R_2 of $G(P)$ from R_1 to R_2; in the latter case, we shall direct it from R_2 to R_1.

It remains to be proved that in $G(P)$ with edges thus directed, every directed path of length two completes into a transitive triangle. For this purpose, consider maximal rectangles R_1, R_2, R_3 such that $R_1\cap R_2=\emptyset$, $R_2\cap R_3=\emptyset$ and $c_1<c_2<c_3$, $d_1<d_2<d_3$. It will suffice to prove that $R_1\cap R_3=\emptyset$. Without loss of generality, we may assume $a_1\leqq a_3$. If $R_1\cap R_3\neq\emptyset$ then P contains the nonempty rectangle $[a_3, b_1]\times$ $\times[c_1, d_3]$. Since $[c_2, d_2]\subseteq[c_1, d_3]$ and since R_2 is a *maximal* rectangle contained in P, we must have $[a_2, b_2]\supseteq[a_3, b_1]$. But then all three of our rectangles R_i contain the nonempty rectangle $[a_3, b_1]\times[c_3, d_1]$, contradicting the assumption that $R_1\cap R_2=$ $=R_2\cap R_3=\emptyset$. ∎

The hypothesis of the lemma cannot be relaxed to replacing "semiconvex" by "simply connected": the graph $G(P)$ of the polyomino P shown in Fig. 2 is not a comparability graph.

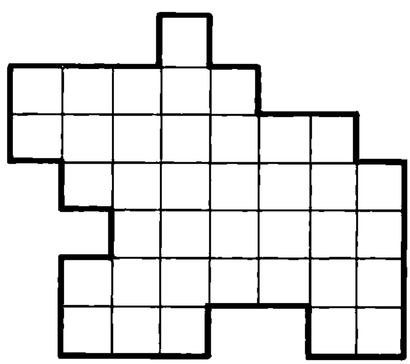

Fig. 2

Problem 3. *Is there a simply connected polyomino P such that $G(P)$ is not a perfect graph?*

Theorem 1. *If P is semiconvex then $\nu(P)=\tau(P)$.*

Proof. If $G=G(P)$ then $\nu(P)$ is the size $\omega(G)$ of the largest clique in G. It is well known and easy to prove that the chromatic number of every comparability graph G equals $\omega(G)$. Hence the family of all maximal rectangles in P may be partitioned into classes S_i $(1\leqq i\leqq\nu(P))$ so that every two rectangles in the same class intersect. But rectangles have the Helly property: if every two of them intersect then the whole family has a nonempty intersection. Hence there are cells c_i $(1\leqq i\leqq\nu(P))$ such that every rectangle in S_i contains c_i. But then $\tau(P)\leqq\nu(P)$ as claimed. ∎

The equality $\nu(P)=\tau(P)$ does not hold in general: Fig. 3 shows a polyomino with $\nu=6$ and $\tau=7$.

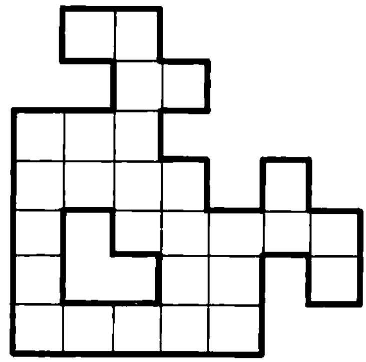

Fig. 3

Problem 4. *Is there a simply connected polyomino with* $\nu \neq \tau$?

Problem 5. *How difficult is the evaluation of* $\nu(P)$ *and/or* $\tau(P)$? *What if the input is restricted to simply connected polyominoes* P?

3. Stable transversals and stochastic functions

A set S of cells in a polyomino P will be called a *stable transversal* if every maximal rectangle in P contains precisely one cell of S. As it turns out, every semiconvex polyomino has a stable transversal. Actually, we shall prove a stronger statement. The *depth* of a cell (x, y) in P is the largest k such that $(x, y-j) \in P$ whenever $0 \leqq j \leqq k$. A *top-cell* in P is a cell $(x, y) \in P$ such that $(x, y+1) \notin P$. A polyomino P will be called *pataconvex* if it has the following property: for every choice of two top-cells (r, d) and (t, d) in the same row of P such that $(s, d) \in P$ whenever $r \leqq s \leqq t$, the depth of each cell (s, d) with $r \leqq s \leqq t$ is at least the minimum of the depths of (r, d) and (t, d).

It is not difficult to see that every semiconvex polyomino is pataconvex. However, the converse is far from being true; as Fig. 4 shows, a pataconvex polyomino may be multiply connected.

Theorem 2. *Every pataconvex polyomino has a stable transversal.*

Proof. We shall describe an easy procedure for finding a certain set S of cells in a pataconvex polyomino P. Then we shall show that every maximal rectangle R contains at most one element of S. Finally, we shall show that every maximal rectangle R contains at least one element of S. To construct S, we consider all the horizontal layers in P one by one: each of these layers splits into pairwise disjoint intervals $I_1, I_2, \ldots, I_k$. Each I_i that contains at least one top-cell of P contributes to S a top-cell of the maximum depth; the remaining intervals I_i contribute no cells at all. Verifying that each rectangle R contains at most one element of S is easy: since each cell in S is a top-cell in P, all the cells in $R \cap S$ come from the top layer of R. Hence these cells come from the same I_i. But I_i contains at most one cell of S. To verify that every maximal rectangle $R = [a, b] \times [c, d]$ contains at least one cell

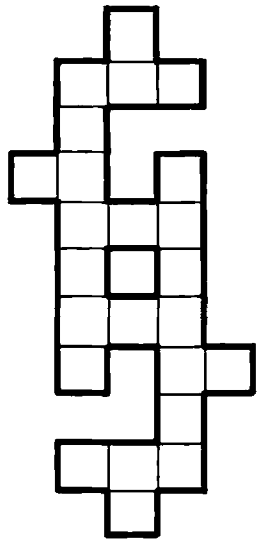

Fig. 4

of S, observe that the top layer of R contains some top-cell (t, d). Hence the interval I_i containing (t, d) contributes precisely one cell, say (r, d), to S. If $a \leqq r \leqq b$ then we are done; otherwise we may assume, without loss of generality, that $r \leqq a$. By the construction of S, the depth of (r, d) is at least the depth of (t, d). Since (r, d) and (t, d) are top-cells, since P is pataconvex and since each (s, d) with $r \leqq s \leqq t$ belongs to P, the depth of each such (s, d) is at least the depth of (t, d). But then P contains the rectangle $[r, b] \times [c, d]$, contradicting the maximality of R. ∎

On the other hand, the polyomino shown in Fig. 5 has no stable transversal. We shall prove a stronger statement. An assignment of nonnegative weights $x_1, x_2, \ldots, x_n$ to the cells $1, 2, \ldots, n$ of a polyomino is called a *stochastic function* if

$$\sum_{i \in R} x_i = 1$$

for every maximal rectangle R. Our aim is to show that the polyomino of Fig. 5 has a unique stochastic function and that this function is fractional valued.

To begin with, let us show that every stochastic function on the polyomino must have

$$(1) \qquad \begin{aligned} x_{26} &= x_{32} = 0 \\ x_{29} &= x_{35} = 0 \\ x_1 &= x_{12} = 0 \\ x_6 &= x_{18} = 0 \\ x_4 &= x_{15} = 0 \end{aligned}$$

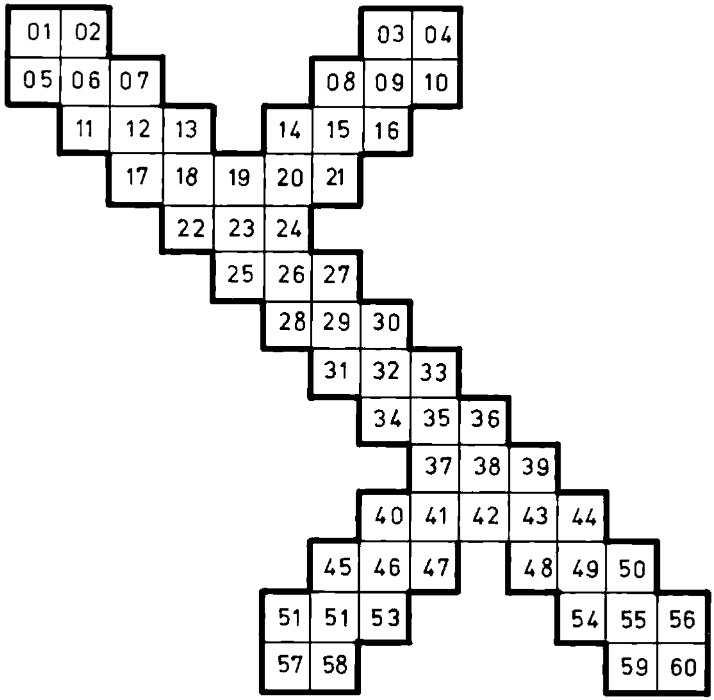

Fig. 5

$$x_9 = x_{20} = 0$$
$$x_{57} = x_{46} = 0$$
$$x_{52} = x_{41} = 0$$
$$x_{60} = x_{49} = 0$$
$$x_{55} = x_{43} = 0.$$

To establish the first line of (1), we note that

$$x_{26}+x_{32} = (x_{26}+x_{27}+x_{28}+x_{29})+(x_{29}+x_{30}+x_{31}+x_{32})$$
$$-(x_{27}+x_{29}+x_{31})-(x_{28}+x_{29}+x_{30}).$$

Since each of the four partial sums equals one, we have $x_{26}+x_{32}=0$; now the desired conclusion follows since x_{26} and x_{32} are nonnegative. The remaining nine lines in (1) are established by analogous arguments. Next, note that

$$x_{19}+x_{20}+x_{24}+2x_{26}+2x_{28}+x_{29}-x_{15}$$
$$= (x_{13}+x_{18}+x_{22})-$$
$$-(x_{12}+x_{13}+x_{17}+x_{18})+$$

$$
\begin{aligned}
&+(x_{19}+x_{20}+x_{23}+x_{24}+x_{25}+x_{26})-\\
&-(x_{18}+x_{19}+x_{20}+x_{22}+x_{23}+x_{24})+\\
&+(x_{26}+x_{27}+x_{28}+x_{29})-\\
&-(x_{25}+x_{26}+x_{27})+\\
&+(x_{17}+x_{18}+x_{19}+x_{20}+x_{21})-\\
&-(x_{14}+x_{15}+x_{20}+x_{21})+\\
&+(x_{14}+x_{20}+x_{24}+x_{26}+x_{28}).
\end{aligned}
$$

Since each of the nine partial sums equals one, we have

$$x_{19}+x_{20}+x_{24}+2x_{26}+2x_{28}+x_{29}-x_{12}-x_5 = 1$$

and, after a substitution from (1),

(2) $$x_{19}+x_{24}+2x_{28} = 1.$$

Relying on the central symmetry of our polyomino, we conclude at once that a similar argument yields

(3) $$x_{42}+x_{37}+2x_{33} = 1.$$

Now we observe that

$$
\begin{aligned}
x_{19}+x_{24}+x_{42}+x_{37} = &(x_{19}+x_{24}+2x_{28})\\
&+(x_{42}+x_{37}+2x_{33})\\
&-2(x_{28}+x_{29}+x_{30})\\
&-2(x_{31}+x_{32}+x_{33})\\
&+2(x_{29}+x_{30}+x_{31}+x_{32}).
\end{aligned}
$$

Using (2) and (3) we find that each of the five partial sums equals one and so

$$x_{19}+x_{24}+x_{42}+x_{37} = 0.$$

Since all the variables are nonnegative, we conclude that

(4) $$x_{19} = x_{24} = x_{42} = x_{37} = 0.$$

Substituting from (4) back into (2) and (3) we obtain

(5) $$x_{28} = x_{33} = 1/2.$$

The rest is easy: (1), (4) and (5) start a chain reaction setting the value of each as yet unspecified x_i at $1/2$. The details may be left to the reader.

Problem 6. *Is there a polyomino with no stochastic function?*

Problem 7. *How difficult is it to decide whether a polyomino has a stable transversal?*

4. End-cells and distinct representatives

A cell in P is called an *end-cell* if it belongs to only one maximal rectangle. The following lemma has been also found independently by C. Christen, P. Duchet, R. L. Rivest and perhaps others.

Lemma. *If a polyomino has at least two cells then it has at least two end-cells.*

Proof. In the top row of P, choose a cell (x, y) of a minimum depth k. Let a be the smallest number such that $(i, y) \in P$ whenever $a \leqq i \leqq x$; let b be the largest number such that $(i, y) \in P$ whenever $x \leqq i \leqq b$. Write $c = y - k$ and $d = y$. It is easy to see that $[a, b] \times [c, d] \subseteqq P$ and that every rectangle in P containing (x, y) is contained in $[a, b] \times [c, d]$. Thus (x, y) is an end-cell. The same argument applied to P turned upside down finds an end-cell in the bottom row of P. This cell is distinct from (x, y) unless P has only one row in which case the assertion is trivial. ∎

Theorem 3. *In every polyomino, the family of maximal rectangles has a system of distinct representatives.*

Proof. By induction on the number of cells. Find an end-cell c. This cell is contained in only one maximal rectangle R; all the remaining maximal rectangles in P continue to be maximal in $P - c$. By the induction hypothesis, these rectangles have a system of distinct representatives; the rectangle R may be represented by c. ∎

An alternative proof of Theorem 3 provides an explicit description of the system of distinct representatives. Let us define left-cells and right-cells by analogy with the definition of top-cells given above. Each maximal rectangle $R = [a, b] \times [c, d]$ contains a left-cell $[a, v]$, a right-cell $[b, w]$ and a top-cell $[x, d]$. Having chosen these three cells, we define $c(R) = (x, y)$ with $y = \min(v, w)$. (Note that $c(R)$ is not always uniquely defined; this ambiguity is easily removed by choosing the largest available values of x, v and w.) We claim that $c(R)$ determines R, and so the cells $c(R_1), c(R_2), \ldots, c(R_M)$ constitute a system of distinct representatives for the family of all maximal rectangles $R_1, R_2, \ldots, R_M$. To justify our claim, we first note that d is the largest number such that $[x, x] \times [y, d] \subseteqq P$. Secondly, we note that $[a, b]$ is a subinterval of the unique maximal interval $[a^*, b^*]$ such that $[a^*, b^*] \times [y, d] \subseteqq P$. Furthermore, if $a^* < a$ then $v < y$, a contradiction. Hence $a = a^*$ and, by a similar argument, $b^* = b$. The rest is trivial: c is the smallest number such that $[a, b] \times$ $\times [c, d] \subseteqq P$.

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